L^∞ NORMS OF HOLOMORPHIC MODULAR FORMS IN THE CASE OF COMPACT QUOTIENT

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ABSTRACT. We prove a sub-convex estimate for the sup-norm of L^2-normalized holomorphic modular forms of weight k on the upper half plane, with respect to the unit group of a quaternion division algebra over $\mathbf{Q}$. More precisely we show that when the L^2 norm of an eigenfunction f is one,

$$\|f\|_\infty \ll k^{\frac{1}{2}-\frac{1}{33}+\varepsilon}$$

for any $\varepsilon > 0$ and for all k sufficiently large.

1. INTRODUCTION

The supremum norm of cusp forms has been a topic of considerable interest in the recent past. Let us first look at the case of holomorphic cusp forms of weight k for the full modular group, $SL(2, \mathbf{Z})$. Let f be such a form. We further assume that f is a Peterson normalised eigenfunction of all the Hecke operators. Then the L^∞ norm of f is by definition the supremum of the bounded $SL(2, \mathbf{Z})$-invariant function $y^{k/2}|f(z)|$:

$$\|f\|_\infty = \sup_{z \in \mathbf{H}} |y^{k/2} f(z)|,$$

where $z = x + iy$ the Poincare upper half-plane $\mathbf{H}$. In [11], H. Xia proved that

$$(1.1) \qquad k^{1/4-\epsilon} \ll \|f\|_\infty \ll k^{\frac{1}{4}+\epsilon} \quad \text{for all} \quad \epsilon > 0.$$

Note that the convexity or 'trivial' bound in this case is $\|f\|_\infty \ll k^{\frac{1}{2}+\epsilon}$.

In the case of Maass forms of weight zero, Iwaniec and Sarnak showed in an important paper ([7]) that

$$\|f\|_\infty \ll \lambda^{\frac{5}{24}+\epsilon} \quad \text{for all} \quad \epsilon > 0,$$

1

where λ is the eigenvalue of f for the hyperbolic Laplacian. Here f has L^2 (or Petersson) norm one. Iwaniec and Sarnak also investigate the supermum norm of eigenfunctions on a compact arithmetic surface. Such a surface is of the form $\Gamma\backslash\mathbf{H}$ where Γ is a cocompact arithmetic subgroup of $SL(2,\mathbf{R})$ arising from quaternion division algebra over $\mathbf{Q}$. They considered the supermum norm of eignefunctions of the Laplacian on $\Gamma\backslash\mathbf{H}$ under the assumption that the eigenfunction in question is also a simultaneous eigenfunction of the Hecke operators $T(n), (n,q) = 1$. Here q is a positive integer depending on the maximal order R in the quaternion division algebra which gives rise to $\Gamma = R(1)$, the group of units of R. The result they prove is the same as in the Maass case, i.e., if f is any such eigenfunction with L^2 norm one then

$$\|f\| \ll \lambda^{\frac{5}{24}+\epsilon} \ \text{ for all } \ \epsilon > 0.$$

The convexity bound here is $\|f\|_\infty \ll \lambda^{\frac{1}{4}+\epsilon}$.

In this note we place ourselves in the same setting as Iwaniec-Sarnak i.e., we consider a cocompact arithmetic subgroup Γ as above. However the functions we consider are holomorphic modular forms for Γ of weight k where k is a positive even integer. Recall that the L^∞ norm is the supermum of the Γ invariant function $y^{k/2}|f(z)|$ if f has weight k. In this situation we prove the following result.

Theorem 1.1. *Let Γ be as above and f a holomorphic modular form for Γ of weight k. Assume that f is a simultaneous eigenfunction of all the Hecke operators. Assume that f has Petersson norm one. Then for all $\varepsilon > 0$ there exist an absolute constant $k_0 > 0$ such that for all $k > k_0$,*

$$\|f\|_\infty \ll k^{\frac{1}{2}-\frac{1}{33}+\varepsilon} \ .$$

The implied constant depends on ε and the group Γ but not on f.

Note that the convexity bound in this case is $\|f\|_\infty \ll k^{\frac{1}{2}}$, which is sharp in some cases if the Hecke assumption is removed; see Remark 2.1 in section 2.4.

In Xia's argument while obtaining (1.1), with both the upper and lower bounds, essential use is made of the presence of a cusp and the Fourier expansion of f in the noncompact case. In fact, this allows him to use Deligne's sharp bound for the Fourier coefficients for the upper bound while taking the point z very high up in the cusp allows for the lower bound. In the setting of our paper, there are no cusps and both of these tools are lost.

Our approach consists in employing the Bergman kernel for the compact quotient $\Gamma\backslash\mathbf{H}$. We embedd f in a orthonormal basis $\{f_j\}$ of the space of modular forms of weight k, each f_j being a simultaneous Hecke eigenform. Recall that the Bergman kernel $h_k(z,w)$ is proportional to $\sum_j f_j(z)\overline{f_j(w)}$. We apply the Hecke operator $T(n)$ in the w-variable and then estimate the resulting function. We first derive a reasonable estimate for $h_k(z,z)$ using some results of Cogdell and Luo ([1]), which is presented in (4.2). We next implement the amplification technique of Iwaniec and Sarnak to highlight the contribution of f and obtain the result.

In particular one does not have a direct $k^{1/4}$ upper bound as in [11], while it is possible that even an upper bound k^ε might hold in Theorem 1.1.

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2. NOTATION AND SETUP

2.1. **Quaternion algebras and orders.** Let $A = \left(\frac{a,b}{\mathbf{Q}}\right)$ be a quaternion division algebra over $\mathbf{Q}$. A has a basis consisting $\{1,\omega,\Omega,\omega\Omega\}$ over $\mathbf{Q}$ and $\omega^2 = a, \Omega^2 = b, \omega\Omega + \Omega\omega = 0$. Here a,b are square-free and we assume that $a > 0$. For details on quaternion algebras, we refer the reader to [2].

Let $\alpha \in A$. We define, as usual, the trace and norm maps by $T(\alpha) = \alpha + \overline{\alpha}$ and $N(\alpha) = \alpha\overline{\alpha}$. Here, $\overline{\alpha}$ is the conjugate to α defined by $\overline{\alpha} = x_0 - x_1\omega - x_2\Omega - x_3\omega\Omega$, if $\alpha = x_0 + x_1\omega + x_2\Omega + x_3\omega\Omega$.

Recall that an order S in A is a subring of A containing 1, finite over $\mathbf{Z}$ and such that S has $\mathbf{Q}$ basis of order 4. Any such order is contained in a maximal order of A. Let R be a maximal order of A and $R(1)$ be its groups of units, i.e., elements of norm 1. Further let

$$R(n) = \{\alpha \in R \mid N(\alpha) = n\}.$$

It is well-known that the $\mathbf{Q}(\sqrt{a})$ algebra $A \otimes_{\mathbf{Q}} \mathbf{Q}(\sqrt{a})$ is split and so there exists an embedding ϕ of A into $M_2(\mathbf{Q}(\sqrt{a}))$ defined by

$$\phi(\alpha) = \begin{bmatrix} \bar{\xi} & \eta \\ b\bar{\eta} & \xi \end{bmatrix},$$

where $\alpha = x_0 + x_1\omega + (x_2 + x_3\omega)\Omega = \zeta + \eta\Omega$. Further it is known that $\det \phi(\alpha) = N(\alpha)$. In the sequel we will work with the image of ϕ in $M_2(\mathbf{Q}(\sqrt{a}))$, so we drop the ϕ from the notation for convenience. Since A is a division algebra, $\Gamma\backslash\mathbf{H}$ is a compact hyperbolic surface. Thus any fundamental domain $\mathcal{F}$ for the action of Γ on $\mathbf{H}$ is compact.

2.2. **Hecke operators.** From the theory of correspondences (see [2]), one can define Hecke operators $T(n)$, $n \geq 1$ using the orbits $R(1)\backslash R(n)$, which are $O(n^{1+\varepsilon})$ in number, as follows. For $f\colon \mathbf{H} \to \mathbf{C}$ holomorphic one defines

$$(2.1) \qquad f \mid T(n) := n^{k/2-1} \sum_{\gamma \in R(1)\backslash R(n)} f \mid_k \gamma,$$

where as usual, we denote

$$f \mid_k \gamma := (\det \gamma)^{k/2}(cz + d)^{-k} f\left(\frac{az+b}{cz+d}\right), \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma.$$

Let us denote the space of modular forms of weight k for Γ by $M_k(\Gamma)$. Analogous to the theory of modular forms for congruence subgroups of the modular group, one knows (see [2]) that there exist an integer q (depending on R) such that for $(n, q) = 1$, the $T(n)$ preserve $M_k(\Gamma)$, are self-adjoint, and satisfy

$$(2.2) \qquad T(m)T(n) = \sum_{d \mid (m,n)} d^{k-1} T(\frac{mn}{d^2}).$$

2.3. **The Bergman kernel.** The Bergman kernel or the reproducing kernel for $\Gamma\backslash\mathbf{H}$ is characterized as the unique function (upto non-zero scalars) $\mathcal{B}(z, w)$ of two variables $z, w \in \mathbf{H}$ (holomorphic in z and anti-holomorphic in w) such that for any holomorphic function f on $\mathbf{H}$, one has (see [9]), writing $w = u + iv$:

$$(2.3) \qquad \int_{\Gamma \backslash \mathbf{H}} v^k f(w) \mathcal{B}(z, w) \frac{dudv}{v^2} = f(z).$$

The Bergman kernel for $M_k(\Gamma)$ can now be written down explicitly as follows. For $n \geq 1$, define the following function:

$$(2.4) \qquad h_k^n(z, w) = \sum_{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in R(n)} n^{k/2} (c\overline{w} + d)^{-k} \left(\frac{z - \gamma \overline{w}}{2i} \right)^{-k}.$$

It is easily checked that $h_k^n(z, w)$ defines a holomorphic function in z and is anti-holomorphic in w. In this case it is well-known (or one can check directly from (2.3), see also [1, 9]) that

$$(2.5) \qquad \mathcal{B}(z, w) = \sum_{j=1}^{d} f_{j,k}(z) \overline{f_{j,k}(w)} = 2^{-1} (k - 1) h_k(z, w),$$

where $f_{j,k}$ is any orthonormal basis of $M_k(\Gamma)$ (which is finite-dimensional) and thus h_k is proportional to $\mathcal{B}$. In our paper we shall take the orthonormal basis to be the one consisting of L^2 normalized Hecke eigenforms.

2.4. **The convexity bound.** The 'convexity bound' can be obtained as an application of Godement's theorem (see [1], [9]) and a calculation in [1]. We record it here for the convenience of the reader.

Note that $\mathcal{B}(z, z) = \frac{k-1}{2} \cdot h_k^1(z, z)$ and writing as in (3.20) adopting the notation and estimates introduced in section 3

$$(2.6) \qquad \|f\|_\infty^2 \leq \frac{k-1}{2} \cdot \max_{\gamma \in \Gamma} |h_\gamma(z)|^{k-4} \cdot \sum_{\gamma \in \Gamma} |h_\gamma(z)|^4 \ll_\Gamma k.$$

and thus $\|f\|_\infty \ll_\Gamma k^{1/2}$.

Remark 2.1. Note that (2.6) holds without assuming that f is a Hecke cusp form, and indeed if one drops the Hecke assumption then this bound $k^{1/2}$ is sharp for

some f. This follows from the Sarnak's multiplicity argument in his letter to Morawetz (see [10]), which shows that for some $f_0 \in M_k(\Gamma)$ one has

$$\|f_0\|_\infty^2 \cdot \mathrm{vol}(\Gamma\backslash\mathbf{H}) \geq \dim M_k(\Gamma) \approx k.$$

3. Estimation of the Bergman kernel

In this section we carry out estimates for the Hecke-transformed Bergman kernel $h_k^n(z, w)$ in terms of n and the imaginary parts of z, w. First we estimate it crudely, using the estimate for $h_k(z, w)$ as in [9], and then use this in conjunction with a trick due to Cogdell-Luo in [1] to arrive at a reasonable estimate for $h_k^n(z, w)$.

We recall *Godement's theorem* on the estimate for the majorant of $h_k(z, w)$ obtained by putting absolute values on its summands. Namely from [9, p. 79, Prop. 2 (iii)] we obtain that (keeping in mind that Klingen's argument holds for any discrete subgroup of $SL(2, \mathbf{R})$, see [9, p. 81])

$$\text{(3.1)} \qquad \lceil h_k(z, w) \rceil \leq \alpha(K)(k-1)^{-1}\mathrm{Im}(w)^{-k/2},$$

where $\lceil h \rceil$ signifies that the absolute values of the summands appearing in the definition of h are considered. Here $w \in \mathbf{H}$, $z \in K$, and for any compact set $K \subset \mathbf{H}$ and $\alpha(K)$ is a constant depending only on K. We now note the following expression of $h_k^n(z, w)$ in terms of Hecke operators: $n^{k/2-1}h_k^n(z, w) = h_k(z, w)\mid_{(w)} T(n)$, i.e.,

$$n^{k/2-1}h_k^n(z, w) = n^{k/2-1} \sum_{\gamma \in R(1)\backslash R(n)} h_k(z, w)\mid_{(w),k} \gamma$$

where the subscript (w) denotes the variable on which the action is considered. Then using (3.1), we easily arrive at the following estimate:

$$\lceil h_k^n(z, w) \rceil \leq \alpha(\mathcal{F})(k-1)^{-1}n^{k/2}|j(\gamma, w)|^{-k} \sum_{\gamma \in R(1)\backslash R(n)} \mathrm{Im}(\gamma w)^{-k/2}$$

$$\text{(3.2)} \qquad\qquad \leq \alpha(\mathcal{F})(k-1)^{-1}n^{1+\varepsilon}\,\mathrm{Im}(w)^{-k/2};$$

where we have taken the compact set K to be the fundamental domain $\mathcal{F}$.

After this preliminary estimate, we now turn to a more refined estimate for $h_k^n(z, w)$. To this end, define, following [1]:

$$h_\gamma(z) = \frac{y}{(z - \gamma\bar{z})/2i \cdot (c\bar{z} + d)}, \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R(n), y = \text{Im}(z).$$

Then

$$(3.3) \qquad y^k h_k^n(z, z) = n^{k/2} \sum_{\gamma \in R(n)} h_\gamma(z)^k = n^{k/2} \Big(\sum_{\gamma: \, |\gamma z - z| \le \delta} + \sum_{\gamma: \, |\gamma z - z| > \delta} \Big),$$

where $0 < \delta < 1$ will be chosen later and we call the first and second terms I and II respectively. In I, we use the estimate

$$(3.4) \qquad |h_\gamma(z)| \le \frac{2y}{(y + \frac{ny}{|cz+d|^2})|cz + d|} \le \frac{1}{n^{1/2}},$$

coupled with the following lemma:

Lemma 3.1. *For $0 < \delta < 1$ small enough and $z \in \mathcal{F}$,*

$$\#\{\gamma \in R(n) \colon |\gamma z - z| \le \delta\} \le n^\varepsilon(n\delta^{1/4} + 1).$$

Proof. We will proceed as in [7]. Namely, we consider the stabilizer of z in $SL(2, \mathbf{R})$ and call it K_z. It is a maximal compact subgroup of $\mathbf{H}$ and thus conjugate to $SO(2, \mathbf{R})$ by a matrix, say, $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$. We next recall the Iwasawa decomposition in $SL(2, \mathbf{R})$ with respect to K_z:

$$(3.5) \qquad SL(2, \mathbf{R}) = N_z A_z K_z; \quad \gamma = nak.$$

First we would assume that $z = i$, and work with the standard Iwasawa decomposition with respect to the standard maximal compact subgroup $K = SO(2, \mathbf{R})$. Here we have a canonical expression for N, A:

$$N = \begin{pmatrix} 1 & \alpha' \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} \beta' & 0 \\ 0 & \beta'^{-1} \end{pmatrix}, \quad (\alpha' \in \mathbf{R}, \beta' \in \mathbf{R}^\times).$$

Let $\gamma' \in SL(2, \mathbf{R})$. Clearly, with the Iwasawa decomposition of γ' and α', β' as above, one has $|\gamma'i - i| = |\alpha' + (\beta'^2 - 1)i| = (\alpha'^2 + (\beta'^2 - 1)^2)^{1/2}$. Thus $|\gamma'i - i| \leq \eta$ implies that

$$|\alpha'| \leq \eta, \quad (1 - \eta)^{1/2} \leq |\beta'| \leq (1 + \eta)^{1/2}.$$

From these we also get $|\beta'|^{-1} \leq 1 + c_1\eta^{1/2}$ for some absolute constant $c_1 > 1$ and η small enough. Thus the above inequalities show that for η small enough, $\|p_i - I\| \ll \eta^{1/2}$; where $p_i = n_ia_i$ from the decomposition (3.5) with respect to $z = i$. This implies, after multiplying by $SO(2, \mathbf{R})$ that

$$(3.6) \qquad\qquad\qquad \gamma' = k + O(\eta^{1/2}).$$

Now we can start from $\gamma \in R(n)$ such that $|\gamma z - z| \leq \delta$ and note that

$$K_z = \gamma_0 K \gamma_0^{-1}, \quad \text{where } \gamma_0 = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix}.$$

Define $\gamma_1 := \gamma_0^{-1}\gamma\gamma_0$. From the inequalitiy:

$$|\gamma_0 Z - \gamma_0 W| \leq a \Rightarrow |Z - W| \leq a/y,$$

for $a > 0$, we find that

$$|(\gamma/n^{1/2})z - z| = |\gamma z - z| \leq \delta \Rightarrow |\gamma_1 i - i| \leq \delta/y \leq c_2\delta =: \eta,$$

for some constant $c_2 > 0$ depending only on Γ. Thus after conjugating (3.6) with γ_0 and η defined as above:

$$(3.7) \qquad\qquad\qquad \gamma/n^{1/2} = k_z + O(\delta^{1/2}),$$

since the entries of γ_0 are bounded by some constant depending only on Γ. Now we can proceed as in [7] by following the description of K_z given there. We start with the quadratic form associated to z:

$$\alpha z^2 + \beta z + \gamma = 0, \quad \alpha, \beta, \gamma \ll 1$$

where α, β, γ are real. We allow ourselves to use the notation γ both for a matrix and a real number in order to be consistent with the notation in [7], but will remind the reader in case of any possibility of confusion. The α, β, γ satisfy [1]

$$(3.8) \qquad\qquad \beta^2 - 4\alpha\gamma = -1.$$

From the explicit description of K_z (see [7, eq. 1.12]), we find that

$$(3.9) \qquad K_z = \left\{ \begin{bmatrix} (t - \beta u)/2 & -\gamma u \\ \alpha u & (t + \beta u)/2 \end{bmatrix} \mid t^2 + u^2 = 4. \right\}$$

Thus, from the canonical description of $\gamma \in R(n)$ (see [7, eq. 1.14]):

$$(3.10) \qquad\qquad \gamma = \begin{bmatrix} x_0 - x_1\sqrt{a} & x_2 + x_3\sqrt{a} \\ bx_2 - bx_3\sqrt{a} & x_0 + x_1\sqrt{a} \end{bmatrix}$$

we get by comparing both sides of (3.7), using the descriptions in (3.9) and (3.10) that [2]

$$(3.11) \qquad\qquad 2x_0/n^{1/2} = t + O(\delta^{1/2}).$$

$$(3.12) \qquad\qquad 2x_1\sqrt{a}/n^{1/2} = \beta u + O(\delta^{1/2})$$

$$(3.13) \qquad\qquad 2x_2/n^{1/2} = -(\gamma - \alpha/b)u + O(\delta^{1/2})$$

$$(3.14) \qquad\qquad 2x_3\sqrt{a}/n^{1/2} = -(\gamma + \alpha/b)u + O(\delta^{1/2}).$$

Also, taking (3.8) into account one obtains that

$$(\gamma + \alpha/b)^2 = (\gamma - \alpha/b)^2 + (1 + \beta^2)/b,$$

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[1]One knows that $\alpha = y^{-1}/2, \beta = -xy^{-1}, \gamma = x^2 y^{-1}/2 + y/2$, in fact the quadratic form for z is obtained by acting the matrix γ_0^{-1} on the polynomial $X^2 + Y^2$.

[2]Note that there is a typo in these equations in [7].

which shows that either $|\gamma + \alpha/b| \geq 1/|b|$ or $|\gamma - \alpha/b| \geq 1/|b|$. Thus one of these quantities is bounded below uniformly for all $z \in \mathcal{F}$ (depending on the sign of b only).

First suppose that $b > 0$. Then we have $|\gamma - \alpha/b| \geq 1/|b|$. The proof now follows that in [7] and we obtain that

$$(3.15) \qquad 4 = t^2 + u^2 = \frac{4x_0^2}{n} + \frac{4ax_3^2}{n(\gamma + \alpha/b)^2} + O(\delta^{1/2}).$$

Taking into account [7, Lemma 1.4] we find

$$(3.16) \qquad \#\{x_0, x_3 \colon |x_0^2 + \frac{ax_3^2}{(\gamma + \alpha/b)^2} - n| \ll n\delta^{1/2}\} \leq n^\varepsilon(n\delta^{1/4} + 1)$$

We have the standard estimate

$$(3.17) \qquad \#\{r, s \colon qr^2 + ps^2 = m; \ q \geq 1, p \geq 0\} \ll m^\varepsilon,$$

see the proof of [7, Lemma 1.4] for example.

Recall that for fixed x_0, x_3, both of which are $\ll n^{1/2}$, (this follows directly from (3.7), (3.11), (3.12), (3.13) and (3.14)) the number of integral solutions x_1, x_2 (both of which are $\ll n^{1/2}$ by the same reason as above) is, by (3.17)

$$(3.18) \quad \#\{x_1, x_2 \ll n^{1/2} \colon ax_1^2 + bx_2^2 = x_0^2 + abx_3^2 - n\} \ll |x_0^2 + abx_3^2 - n|^\varepsilon \ll n^\varepsilon.$$

Thus combining (3.16) and (3.18) we see finally

$$\#\{\gamma \in R(n) \colon |\gamma z - z| \leq \delta\} \ll n^\varepsilon(n\delta^{1/4} + 1).$$

This settles the case $b > 0$. When $b < 0$ our choice would be $|\gamma - \alpha/b| \geq 1/|b|$, and this case is completely similar to the previous one. This completes the proof of the lemma. $\qquad \square$

We are now in a position to derive a reasonable estimate for $h_k^n(z, z)$. Let us go back to (3.3). For the sum I, we get the following estimate:

$$(3.19) \qquad \sum_{\gamma:\, |\gamma z - z| \leq \delta} |h_\gamma(z)|^k \leq n^{-k/2} \cdot n^\varepsilon (n\delta^{1/4} + 1).$$

For the sum II, we do the following:

$$(3.20) \qquad \sum_{\gamma:\, |\gamma z - z| > \delta} |h_\gamma(z)|^k \leq \Big(\max_{\gamma:\, |\gamma z - z| > \delta} |h_\gamma(z)|^{k-k_0}\Big) \cdot \sum_\gamma |h_\gamma(z)|^{k_0},$$

where $k_0 > 2$ is a positive integer to be chosen later and use the estimate (3.2) for $\sum_\gamma |h_\gamma(z)|^{k_0} = \lceil h_{k_0}^n(z, z) \rceil$. Next, [1, Lemma 1] shows that

$$(3.21) \qquad |h_\gamma(z)| \leq (1 + \delta^2)^{-1/2}, \quad \text{if} \quad |\gamma z - z| > \delta.$$

We remind the reader that it is easy to see that [1, Lemma 1] holds for all $\gamma \in GL_2^+(\mathbf{R})$ with $\det \gamma \geq 1$. We put (3.21) in (3.20). Thus from (3.3), using (3.19) and (3.20) together we have,

$$y^k |h_k^n(z, z)| \ll n^\varepsilon (n\delta^{1/4} + 1) + O\left(n^{1+\varepsilon}(1 + \delta^2)^{-(k-k_0)/2}\right).$$

We are now in a position to prove Theorem 1.1.

4. Proof of Theorem 1.1

In this section we will prove Theorem 1.1. First we choose a value for δ which gives rise to a decay in terms of k in the sum II in the $h_k^n(z, z)$. We also note that δ will depend on n (used in estimating $h_k^n(z, z)$) but we suppress it in notation for convenience. We use the results of the previous section along with the amplification technique of [7] to get our result.

Proof. To begin with, let us choose

$$(4.1) \qquad\qquad\qquad\qquad\qquad \delta := \frac{C}{n^\beta};$$

where C is a sufficiently small positive constant depending only on the group Γ such that Lemma 3.1 holds and $\beta > 0$ would be chosen later. The estimate for $h_k^n(z, z)$ now reads:

$$(4.2) \qquad y^k |h_k^n(z, z)| \ll n^{1-\beta/4+\varepsilon} + O\left(n^{1+\varepsilon}(1 + Cn^{-2\beta})^{-(k-k_0)/2}\right).$$

We define the 'normalized' eigenvalues for each $1 \le j \le d$,

$$(4.3) \qquad\qquad\qquad\qquad \eta_j(n) := \lambda_j(n)/n^{(k-1)/2};$$

and then the Hecke relation takes the form

$$(4.4) \qquad\qquad\qquad \eta_j(m)\eta_j(n) = \sum_{d|(m,n)} \eta_j(mn/d^2).$$

We start from the equalities

$$\sum_{j=1}^d y^k f_j(z)\overline{f_j(w)} \sum_{n \le N} |\alpha_n \eta_j(n)^2| = y^k \sum_{m,n} \alpha_n \overline{\alpha_m} \sum_j f_j(z)\overline{f_j(w)}\eta_j(m)\eta_j(n)$$

$$= \sum_{m,n} \alpha_n \overline{\alpha_m} \sum_{d|(m,n)} \eta_j(mn/d^2) \sum_j y^k f_j(z)\overline{f_j(w)}$$

$$(4.5) \qquad\qquad\qquad = \sum_{m,n} \alpha_n \overline{\alpha_m} \sum_{d|(m,n)} \frac{d}{(mn)^{1/2}} y^k h_k^{\frac{mn}{d^2}}(z, w)$$

Using the inequality

$$(1 + Cn^{-2\beta})^{-(k-k_0)/2} \ll \frac{n^{2M\beta}}{\binom{k-k_0}{M}}, \quad k - k_0 \gg M$$

in (4.2) and putting the estimate (4.2) for $h_k^n(z, z)$ in (4.5), we get

$$\sum_{j=1}^{d} y^k f_j(z)\overline{f_j(w)}\sum_{n\leq N}|\alpha_n\eta_j(n)^2|$$

$$\ll k\left(\sum_{m,n\leq N}|\alpha_n||\alpha_m|\sum_{d|(m,n)}\frac{d}{(mn)^{1/2}}\left((\frac{mn}{d^2})^{1-\beta/4+\varepsilon}+\frac{(\frac{mn}{d^2})^{1+2M\beta+\varepsilon}}{\binom{k}{M}}\right)\right)$$

$$(4.6)\quad \ll k\left(N^\varepsilon(\sum_{n\leq N}|\alpha_n|^2)+\frac{N^{1+8M}}{\binom{k}{M}}(\sum_{n\leq N}|\alpha_n|^2)\right),$$

here we have chosen $\beta = 2-\varepsilon'$ for suitable ε' and have used the AM-GM inequality on $|\alpha_m|$ and $|\alpha_n|$ to arrive at (4.6). We now would use the amplification method to arrive at an estimate of the sup-norm as follows. Let us fix an eigenform f_{j_0}. The choice for α_n is the same as in [7], namely

$$(4.7)\qquad \alpha_n = \begin{cases} \eta_{j_0}(p) & \text{if } n = p \leq N^{1/2} \\ -1 & \text{if } n = p^2 \leq N \\ 0 & \text{otherwise .} \end{cases}$$

Recall that under the Jacquet-Langlands correspondence [8, p. check] and also [5], there exists a cusp form F_{j_0} of weight k on $\Gamma_0(D)$ with D depending only on the order R such that the Hecke eigenvalues of F_{j_0} coincide with those of f_{j_0} for all $(n, q) = 1$. Thus Deligne's bound holds for $\eta_{j_0}(n)$ for $(n, q) = 1$.

Keeping in mind the Hecke relation for primes $p \nmid q$:

$$\eta_{j_0}^2(p) - \eta_{j_0}(p^2) = 1,$$

and using Deligne's bound and (4.7), we conclude from (4.6), that

$$\|f_{j_0}\|_\infty^2 \left(\sum_{p\leq N^{1/2},p\nmid q} 1\right)^2 \ll kN^{1/2+\varepsilon} + \frac{kN^{3/2+8M+\varepsilon}}{\binom{k-k_0}{M}} \ll kN^{1/2+\varepsilon} + \frac{N^{3/2+8M+\varepsilon}}{k^{M-1}}.$$

Here we have chosen M to be absolutely bounded. We obtain finally

$$\|f_{j_0}\|_\infty^2 \ll kN^{-1/2+\varepsilon} + \frac{N^{1/2+8M+\varepsilon}}{k^{M-1}}.$$

We choose N by

$$N = k^{\frac{M}{1+8M+\varepsilon}}$$

to obtain for large k:

$$\|f_{j_0}\|_\infty^2 \ll k^{1-\frac{M/2}{1+8M+\varepsilon}+\varepsilon} \quad \text{or,} \quad \|f_{j_0}\|_\infty \ll k^{\frac{1}{2}-\frac{M/4}{1+8M}+\varepsilon}.$$

We choose $M = 4$ and this completes the proof of Theorem 1.1. $\square$

Remark 4.1. Clearly, the bound improves as M increases. However the rate of improvement is negligible. For example, $M = 4$ produces the exponent .4697 in Theorem 1.1, whereas $M = 100$ produces the exponent .46879.

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